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Magnon localization in Mattis glass

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Abstract
We study the spectral and transport properties of magnons in a model of a disordered magnet called Mattis glass, at vanishing average magnetization. We find that in two-dimensional space, the magnons are localized with the localization length which diverges as a power of frequency at small frequencies. In three-dimensional space, the long wavelength magnons are delocalized. In the delocalized regime in $3d$ (and also in $2d$ in a box whose size is smaller than the relevant localization length scale) the magnons move diffusively. The diffusion constant diverges at small frequencies. However, the divergence is slow enough so that the thermal conductivity of a Mattis glass is finite, and we evaluate it in this paper. This situation can be contrasted with that of phonons in structural glasses whose contribution to thermal conductivity is known to diverge (when inelastic scattering is neglected).

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1. Introduction

For two reasons, the study of low energy excitations in spin glasses—spin glass *magnons*—is met with notorious difficulties: first, the energetic frustration characteristic for glasses implies the existence of many nearly degenerate minima, i.e. the conceptual status of ‘small’ fluctuations forming on top of any one of those configurations remains somewhat dubious. Second, even if a sufficiently inert extremal configuration was known, the solution of the appropriately linearized problem would still pose a highly nontrivial problem.

To get the problem at least partially under control, magnons in spin glasses are commonly described in the language of mean field theory [1, 2]. Within this approach one finds that the dispersion of the excitations forming on top of a background of vanishing average magnetization

$$\omega(p) \propto |p|, \tag{1}$$

is linear in analogy to the spinwave dispersion of antiferromagnets. Here, ω and p denote frequency and wave vector of the excitations, respectively. Mean field theory further predicts that magnons in a glass with nonzero average magnetization m have a ferromagnet-like branch of excitations,

$$\omega(p) \propto p^2. \tag{2}$$

However, in view of the difficulties alluded to above, there is no reason *a priori* why mean field theory qualifies to describe spin glasses at all. For example, it has been known for more than a decade that mean field theory breaks down below the critical dimension $d_c = 2$ (see [3–5]). (Yet above the critical dimension arguments can be given which support the mean field theory results equations (1) and (2) for the excitation spectrum.)

Furthermore, an important question which mean field theory completely fails to address is the transport properties of magnons: while excitations in conventional magnetic materials propagate ballistically, the disorder inherent to spin glasses renders the dynamics of magnons diffusive and, eventually, leads to mechanisms of localization. Clearly, these phenomena cannot be described in terms of a spatially uniform mean field[4].

It is the main objective of the present paper to introduce an alternative approach to the problem which is not burdened by these limitations. Developed in close analogy to the field theory approach to electron dynamics in disordered solids, the formalism below can be employed to address both spectral *and* localization properties of magnons. On the other hand, it has nothing to say about the first problem mentioned above, identification and analysis of reference ground states. For this reason, we chose to introduce the approach on a prototypical variant of a spin glass, the so-called Mattis glass, for which this problem simply does not exist.

To prepare the definition of the Mattis glass, let us recall that spin glasses [7] are usually described by the exchange Hamiltonian

$$H = -\frac{1}{2}\sum_{ij\alpha} J_{ij} S_i^\alpha S_j^\alpha, \tag{3}$$

where i, j refer to nearest-neighbour sites on some lattice, J_{ij} are random exchange constants and $\hat{S}^\alpha$, $\alpha = 1, 2, 3$ is a vector representing the spin. (In this paper we assume that $|\mathbf{S}| \equiv S \gg 1$ is sufficiently large so that the spin system can be treated as classical.) The spin equations of motion are then given by,

$$\frac{\partial S}{\partial t} = [H, S], \qquad \left[S_j^\alpha, S_k^\beta\right] = \mathrm{i}\delta_{jk}\sum_\gamma \epsilon^{\alpha\beta\gamma} S_k^\gamma, \tag{4}$$

where $\epsilon^{\alpha\beta\gamma}$ is the usual antisymmetric tensor.

As mentioned above, one of the major difficulties hampering analytical progress on the problem posed by (4) is that the ground state(s) of the Hamiltonian are not known. For this reason, we consider here a simplified variant of (3), the so-called Mattis glass. The Mattis glass is defined by,

$$H = -\frac{J}{S}\sum_{jk\alpha} \xi_j\xi_k S_j^\alpha S_k^\alpha, \tag{5}$$

i.e. a spin-glass Hamiltonian for which the exchange constants $J_{jk} \sim \xi_j\xi_k$ *factorize*. Here, J is a constant parameter setting the energy units, j, k are nearest-neighbour sites of a d-dimensional lattice, and ξ_j are bimodally distributed random variables taking value 1 with

[4] Mean field theory becomes exact for spin glasses with infinite range interactions. In that case, one finds [6] that the frequency dependent density of magnon modes (for a precise definition of this quantity, see equation (7) below) scales as $\rho(\omega) \sim \omega^{3/2}$. However, this power-law is at variance with the behaviour observed for 'real' spin glasses.

probability p and -1 with probability $1 - p$. This factorization entails that a degenerate family of spin configurations minimizing the Hamiltonian can readily be written down:

$$S_i^{\alpha(0)} = S \operatorname{sign}(\xi_i)\, n^\alpha, \tag{6}$$

where n is a unit vector with arbitrary direction (due to rotational invariance of equation (5). Thus the Mattis glass is a model with a disordered ground state but without the frustration inherent to usual spin systems. To understand the physical features of Mattis glass magnons we need to explore the equations of motion (4), linearized around (6). (Under the presumed condition $S \gg 1$ anharmonic fluctuations, i.e. magnon *interactions* can safely be neglected.) We emphasize that this part of the analysis is of similar complexity to the study of magnons in 'real' spin glasses.

The first to study magnon propagation in the Mattis glass was David Sherrington. In a series of papers [8] he found the spectral density of magnons in the Mattis glass and their scattering length. Additionally, [3] is an important work studying magnons in a 1D Mattis glass. It has to be mentioned that any random one-dimensional spin chain with nearest-neighbour interaction is automatically a Mattis glass, and since in 1D the scattering length coincides with the localization length all the transport properties of Mattis magnons in 1D are known [3, 8]. Because of that, in this paper we will concentrate on dimensionalities higher than one. Before turning to the methodological aspects of the analysis, let us summarize our main results and relate them to earlier work on similar problems.

Summary of results. In this paper we will characterize the behaviour of magnons in terms of their dispersion relation $\omega(p)$, the frequency dependent localization length, $l(\omega)$ and the so-called density of frequencies

$$\rho(\omega) = \frac{1}{N} \sum_{n=1}^{N} \delta(\omega - \omega_n), \tag{7}$$

where ω_n are the frequencies of the magnon modes and N is the total number of modes in the glass. Borrowing terminology from the physics of disordered electron systems, we will henceforth refer to $\rho(\omega)$ as the density of states (DoS). In a way to be formulated precisely below, the localization length, $l(\omega)$, is a measure for the exponential decay of magnon modes at frequency ω. As measurable observables related to these quantities we will consider the thermal conductance of the glass and its specific heat.

We first note (see the discussion in the end of the next section) that if the average magnetization of a Mattis glass is nonzero, the behaviour of its magnons at very low frequencies closely resembles that of phonons in a *structural* glass. Since phonons in structural glasses have already been discussed in [9], in this paper we concentrate mostly on the Mattis glass with zero average magnetization, or with $p = 1/2$. In some instances below where we do consider $p \neq 1/2$, this will be explicitly stated.

For the dispersion relation of the un-magnetized Mattis glass we find

$$\begin{aligned} &\mathrm{Re}\,\omega \propto p, && \mathrm{Im}\,\omega \propto p^2 && \text{in } 3d, \\ &\mathrm{Re}\,\omega \propto \frac{p}{\sqrt{\log\left(\frac{\Lambda}{p}\right)}}, && \mathrm{Im}\,\omega \propto \frac{p}{\log\left(\frac{\Lambda}{p}\right)^{\frac{3}{2}}}, && \text{in } 2d. \end{aligned} \tag{8}$$

Here, Λ is a momentum cutoff, reciprocal to the minimal wavelength of magnons.

This result is consistent with the discussion in references [5] and [4] where it is argued that $\mathrm{Im}\,\omega \propto p^{d-1}$ above the critical dimension $d > d_c = 2$. This behaviour can be contrasted with the behaviour of phonons in structural glasses whose dispersion relation is

$$\mathrm{Re}\,\omega \propto p, \qquad \mathrm{Im}\,\omega \propto p^{d+1}. \tag{9}$$

Equation (9) is often interpreted as a manifestation of Raleigh scattering. A closely related formula also holds for glasses with nonzero average magnetization M. It reads

$$\mathrm{Re}\,\omega \propto p^2, \qquad \mathrm{Im}\,\omega \propto p^{d+2}. \tag{10}$$

Its relation with to equation (9) will be discussed at the end of this paper.

In [8] self consistent diagrammatic methods were applied to derive the DoS

$$\begin{aligned} \rho(\omega) &\propto \omega|\log(\omega)| && \text{in } 2d, \\ \rho(\omega) &\propto \omega^2 && \text{in } 3d. \end{aligned} \tag{11}$$

Our field theoretical analysis below will confirm this result. Also notice that equations (11) coincide with the DoS deduced from mean field theory equation (1) (up to a logarithmic prefactor in $2d$).

The central result of this paper regards the localization properties of magnons. We find that in $2d$ magnons are localized on the frequency dependent scale

$$l(\omega) \propto \omega^{-\frac{1}{16\pi}}. \tag{12}$$

(In contrast, phonons in structural glasses or, equivalently, magnons in Mattis glass with nonzero M, are subject to a weaker localization mechanism leading to an exponentially diverging localization length [9].) At length scales below the localization length the magnons move diffusively. In the $3d$ case we find that the dynamics is diffusive no matter how small the frequency, i.e. there is no localization.

In both $2d$ and $3d$, the diffusion constant diverges as the frequency of magnons goes to zero:

$$D(\omega) \propto \frac{1}{\omega^{d-1}}. \tag{13}$$

(Compare this with $D(\omega) \propto \omega^{-d-1}$ for structural glasses [9]). Dispersive observables such as the thermal conductivity are related to the product of diffusion constant and density of states, $\rho(\omega)D(\omega)$ (weighted by the specific heat, $C(\omega)$, and integrated over frequency.) A glance at equation (11) shows that this quantity is an integrable function at small ω. As an important consequence, we find that, independent of dimensionality, the thermal conductivity of magnons in a Mattis glass is finite. Specifically, for $2d$,

$$\kappa = \frac{k^2 T}{12\hbar} \log\left(\frac{4\pi\Lambda^2 J^2}{\alpha k^2 T^2}\right). \tag{14}$$

Here α measures the correlation volume of spins (often $\alpha \propto \Lambda^{-d}$ for short range correlated spins). Note that the conductivity is only weakly disorder dependent. This is a consequence of the fact that the diffusion constant/DoS scales linearly/inversely linear with the disorder strength. (Similar behaviour is observed, e.g., for quasiparticles in high temperature superconductors [10].)

In $3d$, the thermal conductivity is given by the fully disorder-independent result,

$$\kappa = \frac{\Lambda}{9\pi}\frac{k^2 T}{\hbar}. \tag{15}$$

Owing to the absence of localization, the thermal *conductance* is obtained from (15) by a trivial multiplication by the system size: Ohm's law.

In this paper we also discuss the Mattis glass with nonzero magnetization. The magnon's low frequency behaviour is closely related to that of phonons in a glass. Thus the density of states is $\rho(\omega) \propto \omega^{\frac{d-2}{2}}$, in agreement with equation (2). The thermal conductivity is infinite because of the weak scattering at small frequencies, as can be seen from the frequency dependent diffusion constant $D(\omega) \propto 1/\omega^{\frac{d}{2}}$. Finally, at frequencies higher than a certain frequency ω_c, the magnons' behaviour crosses over to that of a glass with vanishing magnetization. In $2d$, $\omega_c = JM/\alpha$, where J is the spin–spin interaction constant multiplied by the square of the lattice constant and α is the correlation volume of spins, see below. In $3d$, $\omega_c = JM/(\alpha\Lambda)$, where Λ is the inverse lattice spacing.

This concludes our preliminary summary of results. The rest of the paper is organized as follows. In the next section we derive the magnon equations of motion for Mattis glass. We then formulate the field theory approach (section 3) and use it to discuss magnon localization and transport properties in $2d$ and $3d$ (section 4).

2. The equations of motion

The most straightforward way to introduce magnons on a technical level is by the Holstein–Primakoff transformation [8]. For those spins S_j with $\xi_j > 0$, we write, in the harmonic approximation (site indices suppressed for notational transparency),

$$S^z = S - a^\dagger a, \quad S^+ \approx a\sqrt{2S}, \quad S^- \approx a^\dagger\sqrt{2S}, \tag{16}$$

while for those whose $\xi_j < 0$ we write

$$S^z = -S + a^\dagger a, \quad S^+ \approx a^\dagger\sqrt{2S}, \quad S^- \approx a\sqrt{2S}, \tag{17}$$

where $a, a^\dagger$ are Holstein–Primakoff bosons. Substituting in equation (5) and expanding to second order in $a, a^\dagger$, one obtains the quadratic Hamiltonian [8]

$$\delta H = \frac{J}{2}\sum_{jk=1}^{N}\begin{pmatrix} a_j^\dagger & a_j \end{pmatrix}\begin{pmatrix} h_{jk} & \Gamma_{jk} \\ \Gamma_{jk} & h_{jk} \end{pmatrix}\begin{pmatrix} a_k \\ a_k^\dagger \end{pmatrix} \equiv \frac{J}{2}\psi^\dagger\mathcal{H}\psi, \tag{18}$$

where

$$\mathcal{H} = \begin{pmatrix} h & \Gamma \\ \Gamma & h \end{pmatrix}, \qquad \psi = \begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \tag{19}$$

and h and Γ are $N \times N$ symmetric matrices defined in the following way

$$h_{jj} = 2d, \qquad h_{j,j+\mu} = -\tfrac{1}{2}(1 + \xi_j\xi_{j+\mu}), \qquad \Gamma_{j,j+\mu} = \tfrac{1}{2}(1 - \xi_j\xi_{j+\mu}). \tag{20}$$

Here $j+\mu$ refer to the nearest-neighbour sites of the site j. As discussed in [11], the magnon equation of motion then takes the form

$$\mathcal{H}\psi = \omega\begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_N \end{pmatrix}\psi, \tag{21}$$

where $\mathbb{1}_N$ is the N-dimensional identity matrix. To proceed, it is convenient to perform a unitary rotation

$$\mathcal{H} - \omega\begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_N \end{pmatrix} \quad\to\quad U^\dagger\left(\mathcal{H} - \omega\begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_N \end{pmatrix}\right)U, \qquad \psi \quad\to\quad U^\dagger\psi \tag{22}$$

where the unitary matrix U is defined as

$$U = \frac{1}{2}\begin{pmatrix} \Xi - \mathbb{1}_N & -\Xi - \mathbb{1}_N \\ \Xi + \mathbb{1}_N & \mathbb{1}_N - \Xi \end{pmatrix}, \tag{23}$$

and $\Xi_{jk} = \delta_{jk}\xi_k$. As a result, equation (21) assumes the block-diagonal form

$$\frac{J}{2}\begin{pmatrix} h - \Gamma & 0 \\ 0 & h - \Gamma \end{pmatrix}\psi = \omega\begin{pmatrix} -\Xi & 0 \\ 0 & \Xi \end{pmatrix}\psi, \tag{24}$$

where the disorder-independent combination $h - \Gamma$ is but the d-dimensional lattice Laplacian Δ_{jk}. (As a side remark we mention that the unitary equivalence $\mathcal{H} \sim -\Delta > 0$ states the positivity of the operator $\mathcal{H}$—for a bosonic problem, a necessary stability condition.) Thus we find

$$-\frac{J}{2}\sum_k \Delta_{jk}\psi_k = \pm\omega\xi_j\psi_j. \tag{25}$$

Here the two signs refer to the upper/lower half of the vector ψ. Physically, they correspond to two magnon branches of opposite frequency. In the context of Mattis glass equation (25) first appeared in [3]. As a last step, we take the continuum limit of this equation. Trading the discrete index j for a continuous coordinate x, we arrive at[5]

$$-\frac{J}{2}\Delta\psi = \pm\omega\xi(x)\psi, \tag{26}$$

where Δ is the continuum Laplacian and the discrete random variable ξ_j was replaced by a function $\xi(x)$, randomly taking values +1 or -1 with probabilities p and $1 - p$ at different points in space.

As it is hard to deal with bimodally distributed variables, we are going to replace $\xi(x)$ according to $\xi \to M + V(x)$, where M is the average magnetization per spin $M = 2p - 1$ and $V(x)$ is a random Gaussian variable with zero mean and correlation

$$\langle V(x)V(y)\rangle = \alpha\delta(x - y). \tag{27}$$

Remembering the definition of V as a continuous version of the variables ξ_k, we deduce that the parameter α has the meaning of the correlation volume of spins in the Mattis glass. (For uncorrelated spins, α is of the order of the elementary lattice volume.)

In view of the universality of extended random systems with respect to changes in the microscopic realization of the disorder, we trust that taking a continuum limit and modelling the randomness according to (27) are permissible simplifications. At any rate, the final form of the equations of motion on which our further analysis will be based reads as

$$\left(\frac{J}{2}\Delta + \omega M + \omega V(x)\right)\psi = 0, \tag{28}$$

where we have suppressed the $\pm$ sign in front of the frequency.

Structurally, equation (28) bears a resemblance to the Schrödinger equation of a quantum particle in a random potential. The crucial difference is, however, that what would have been an energy in that equation is now proportional to magnetization, and what would have been the disorder strength is proportional to the frequency of magnons.

Equation (28) is also very similar to the main equation of [9] where phonons in structural glasses were studied. However, what is $M + V(x)$ in equation (28) was the mass density $m(x)$ in [9], and as such, it was a strictly positive quantity. In the Mattis glass, on the other hand, $M + V(x)$ (being a continuum analogue of the bimodal distribution $\xi_j \in \{-1, 1\}$) can and must be negative for some values of x. Yet the methods employed in [9] required to take $M + V(x)$ as a random Gaussian variable centred around $M \equiv \langle m(x)\rangle$. Without discussing whether that would invalidate any of the conclusions of [9] with respect to the structural glasses, we immediately deduce that the properties of magnons in Mattis glasses with nonzero average magnetization M have effectively been already derived, and [9] can be consulted to find out how that was done. (For completeness, the main results had been summarized in the previous section.) In what follows we concentrate on the complementary case of the Mattis glass with

[5] After taking the continuum limit, J now becomes the spin–spin interaction constant times the square of the lattice spacing.

vanishing magnetization $M = 0$ (or $p = 1/2$), with the exception of section 4, where $M \neq 0$ will also be considered.

3. Field theory

3.1. Green functions

All relevant information about the eigenvalue problem defined by (21) and (28) is contained in the advanced and retarded Green functions

$$G^{\pm} = \frac{1}{\omega + \frac{J}{2} V^{-1} \Delta \pm \mathrm{i}\epsilon}, \tag{29}$$

where $\epsilon > 0$ is infinitesimal and we have set the average magnetization to zero, $M = 0$. An equivalent (see the appendix), but for our purposes more convenient representation of the Green function reads as

$$G^{\pm} = \left[\omega V + \frac{J}{2} \Delta \pm \mathrm{i}\epsilon \,\mathrm{sign}\, \omega \right]^{-1} V. \tag{30}$$

The two quantities we shall focus on in the following are

$$\begin{aligned} &C_1(\omega) \equiv \langle G^{+}(\omega; \mathbf{x}, \mathbf{x}) \rangle, \\ &C_2(\omega, \Omega; |\mathbf{x} - \mathbf{y}|) \equiv \langle G^{+}(\omega + \Omega/2; \mathbf{x}, \mathbf{y}) G^{-}(\omega - \Omega/2; \mathbf{y}, \mathbf{x}) \rangle, \end{aligned} \tag{31}$$

where the averaging $\langle \ldots \rangle$ is over 'disorder' V. As usual, the average DoS can be represented in terms of the Green functions as

$$\rho(\omega) = -\frac{1}{\pi} \mathrm{Im} C_1(\omega). \tag{32}$$

$C_2(\omega, \mathbf{p})$ can be used to calculate the diffusion constant of magnons. At small $\Omega \ll \omega$ and small $\mathbf{p}$,

$$C_2(\omega, \Omega; \mathbf{p}) \approx \frac{4\pi \rho(\omega)}{D(\omega)\mathbf{p}^2 + \mathrm{i}\Omega}, \tag{33}$$

where $D(\omega)$ is the diffusion constant of magnons at frequency ω.

The thermal conductivity of the system is then given by [9]

$$\kappa = \int_0^{\infty} \mathrm{d}\omega\, \rho(\omega) C(\omega, T) D(\omega), \tag{34}$$

where C is the specific heat of magnons at frequency ω,

$$C(\omega, T) = \frac{(\hbar\omega)^2}{kT^2 \left(2 \sinh \frac{\hbar\omega}{2kT}\right)^2}. \tag{35}$$

To compute the disorder average involved in the definition of the correlation functions $C_{1,2}$, we employ the field theoretical formalism of the nonlinear σ-model, an approach that has been met with tremendous success in the field of disordered fermion physics. In fact, the application of this field theory to 'glassy' problems is by no means original. In a pioneering work, John, Sompolinsky and Stephen [9] attacked the related problem of phonon localization in structural glasses by the same formalism. However, while at first sight the presence (phonons) or absence (Mattis glass magnons) of the parameter M in (28) may appear to be of minor significance, the opposite is the case. In fact, essential elements of the structure of the theory depend on this point, which is why very different physical results are obtained (cf the discussion above.)

To prepare the averaging over the disorder, we consider the supersymmetric generating functional:

$$\mathcal{Z}[V] = \int D\Psi \exp(-S[\bar{\Psi}, \Psi]), \tag{36}$$

where the action S is defined as

$$S = \mathrm{i}\bar{\Psi}\left[\frac{J}{2}\Delta + \left(\omega + \frac{\Omega}{2}\sigma_3^{AR}\right)V + \mathrm{i}\epsilon\sigma_3^{AR}\right]\Psi. \tag{37}$$

Here, $\Psi \equiv \{\Psi^\alpha\}$, $\alpha = 1, \ldots, 8$ is an eight-component vector field, half of whose components are complex variables, while the remaining components are anticommuting (Grassmann variables). Of the commuting (anticommuting) variables, one half refers to the retarded sector of the theory, the other half to the advanced sector. The simultaneous presence of both sectors is indicated by the presence of the Pauli matrix σ_3^{AR} operating in advanced–retarded space. While our so far counting accounts for four different types of variables (commuting–anticommuting/advanced–retarded), a further doubling of the number of integration variables is required [12] by the time reversal invariance of the problem (technically, the fact that we are dealing with a symmetric operator.)

Referring for a comprehensive introduction to the apparatus of supersymmetry in statistical physics to [12] we here merely mention that the rationale behind introducing a supersymmetric structure is that $\mathcal{Z}[V] = 1$ is automatically unit-normalized. (The operator determinants resulting from the integration over commuting/anticommuting variables, resp., cancel each other.) Not having to worry about determinantal prefactors, the average over the Gaussian distribution of the disorder becomes a straightforward operation. Again referring for a detailed discussion to [12], we here just note that a Hubbard–Stratonovich decoupling of the four-fermion 'interaction' generated by the disorder averaging results in

$$\mathcal{Z} \equiv \langle \mathcal{Z}[V]\rangle = \int \mathcal{D}R \exp(-S[R]), \tag{38}$$

where $R = \{R^{\alpha\beta}\}$ is an eight-dimensional matrix field, whose action reads as

$$S[R] = \frac{1}{4\alpha\omega^2}\int \mathrm{str}(R^2) + \frac{1}{2}\mathrm{str}\ln\mathcal{G}[R]^{-1}. \tag{39}$$

Here, str is the so-called supertrace[6] [12], and $\mathcal{G}$ is defined as

$$\mathcal{G}[R]^{-1} = \frac{J}{2}\Delta + \left(1 + \frac{\Omega}{2\omega}\sigma_3^{AR}\right)R. \tag{40}$$

Equation (38) represents an (admittedly complicated) representation of unity, $\mathcal{Z} = 1$. To make use of the formalism, we need to relate our correlation functions $C_{1,2}$ to the generating functional. Indeed, it is straightforward to verify that the one-point correlation function C_1 is obtained by computing the expression

$$C_1(\omega) = \mathrm{i}\langle\langle S^1(\mathbf{x})S^{1*}(\mathbf{x})\rangle_\psi V(\mathbf{x})\rangle_V,$$

where $\langle\ldots\rangle_\psi$ denotes the functional average over the Gaussian action (37) and S^1 refers to the first component of the commuting sector of the field Ψ. After the disorder average this is equivalent to computing

$$C_1(\omega) = \alpha\omega\langle S^1(\mathbf{x})S^{1*}(\mathbf{x})\bar{\Psi}(\mathbf{x})\Psi(\mathbf{x})\rangle. \tag{41}$$

[6] The supertrace generalizes the standard notion of a trace of a matrix to the case of supermatrices. It is defined as $\mathrm{str}R = \sum_{\alpha=1}^4 R^{\alpha\alpha} - \sum_{\alpha=5}^8 R^{\alpha\alpha}$ where $\alpha = 1, \ldots, 4$ and $\alpha = 5, \ldots, 8$ correspond to the bosonic and fermionic components of Ψ^α.

As for the 'two-point' function C_2, certain care has to be exercised with the sign of the frequency arguments: phenomena like diffusion and localization are observed in the limit of small frequency *difference* $|\Omega| \ll \omega$. Assuming that the offset frequency $\omega > 0$ is positive, the analytic structure (cf equation (30)) of the Green function is thus determined as indicated in (31). Accordingly, the two-point function can be represented as

$$C_2(\omega, \Omega; |\mathbf{x}-\mathbf{y}|) = \alpha^2 \left\langle S^1(\mathbf{x})S^{1*}(\mathbf{y})S^2(\mathbf{y})S^{2*}(\mathbf{x})\bar{\Psi}(\mathbf{x}) \left(\omega + \frac{\Omega}{2}\sigma_3^{AR}\right) \Psi(\mathbf{x}) \right.$$
$$\left. \times \bar{\Psi}(\mathbf{y}) \left(\omega + \frac{\Omega}{2}\sigma_3^{AR}\right) \Psi(\mathbf{y}) \right\rangle. \qquad (42)$$

Here S^1 and S^2 stand for advanced and retarded components of the commuting sectors of the vector Ψ.

To actually evaluate the functional expectation values (41) and (42), we need to reduce the (exact) reformulation of the problem, equations (38) and (39), down to a better manageable effective low energy field theory. This reduction is achieved by a gradient expansion around the spatially uniform stationary phase configurations of the action (39). These reference states are determined by solution of the equation

$$\frac{\delta S[\bar{R}]\big|_{\Omega=0}}{\delta \bar{R}(\mathbf{x})} = 0 \Leftrightarrow \bar{R} = \alpha\omega^2 \left(\frac{1}{2\pi}\right)^{\mathrm{d}} \int \frac{\mathrm{d}^d p}{\frac{J}{2}p^2 - \bar{R}}, \qquad (43)$$

where in the second equality p is momentum. (Owing to $\Omega \ll \omega$, the dependence of the mean field configurations on the frequency mismatch Ω is weak, which is why Ω is neglected in (43).)

The simplest solutions to equation (43) acquire the matrix-diagonal form

$$\bar{R} = q_1 + \mathrm{i}q_2\sigma_3^{\mathrm{AR}},$$

where $q_{1,2}$ are real parameters. The detailed form of $q_{1,2}$ depends on the dimensionality of the problem. To avoid reiterations, we therefore first outline the general (independent of dimensionality) architecture of the theory before discussing the cases $d = 1, 2, 3$ separately.

In the following we will rely upon the fact that q_1/q_2 diverges as ω is taken to zero, in a manner specified below by equations (57) and (67). The significance of this relation becomes clear when we note that, physically, $\bar{R}$ plays the role of a self-consistent Born (SCBA) self-energy of the problem (in diagrammatic language, the self-energy calculated neglecting diagrams with crossing impurity scattering lines.)

Indeed, (43) is structurally equivalent to the familiar form of the SCBA self-energy equation. The statement $q_1 \gg q_2$ thus implies that the energy *shift* acquired by impurity induced virtual transitions from one magnon mode into others exceeds the energy *broadening*, i.e. the instability of magnons in a non-translationally invariant environment. Again alluding to the formal analogy to the Green functions of disordered electrons, $q_1 \sim E_F$ plays the role of the 'Fermi energy' of the problem, and $1/q_2 \sim \tau$ is the analogue of the inverse fermionic scattering time, τ. The fact that $q_1/q_2 \sim E_F\tau \gg 1$ implies that we are working with the analogue of a weakly disordered system. At the same time, the parameter $q_1/q_2 \gg 1$ stabilizes the construction of the low energy field theory to be outlined momentarily.

3.2. Construction of a low energy action

The fact that in the limit $\Omega \to 0$ the action (39) is isotropic in the internal matrix space of the theory implies that $\bar{R}$ is but a single element of an entire manifold of solutions of the saddle point equation. Indeed, any configuration

$$T\bar{R}T^{-1} \equiv q_1 + \mathrm{i}q_2 T\sigma_3^{\mathrm{AR}}T^{-1}$$

Table 1. Comparison of basic (upper part) and derived (lower part) scales of the problem. In the fifth row, ν denotes the DoS per volume and $\Gamma^{(d)}$ is the volume of the d-dimensional unit sphere.

Quantity	Fermions	Magnons
Fermi energy	E_F	q_1
Scattering time	τ	$(2q_2)^{-1}$
Mass	m	J^{-1}
Diffusion constant, D_0	$D_0 = \frac{2E_F\tau}{dm}$	$D_0 = \frac{Jq_1}{dq_2}$
Density of states, ν	$\nu = \frac{\Gamma^{(d)}}{(2\pi)^d}(2mE_F)^{\frac{d-2}{2}} dm$	$\nu = \frac{\Gamma^{(d)}}{(2\pi)^d}\left(\frac{2q_1}{J}\right)^{\frac{d-2}{2}} \frac{d}{J}$

represents another solution. Here $T \in \mathrm{OSp}(4|4)$ is an eight-dimensional supermatrix lying in the supergroup OSp(4|4), i.e. the maximal group manifold compatible with the internal symmetries of the problem [12]. Generalizing to slowly fluctuating rotations, we obtain the matrix field

$$Q(\mathbf{x}) \equiv T(\mathbf{x})\sigma_3^{\mathrm{AR}}T^{-1}(\mathbf{x}) \in \mathrm{OSp}(4|4)/\mathrm{OSp}(2|2) \times \mathrm{OSp}(2|2) \tag{44}$$

as the central degree of freedom of the theory.

The final step in the construction is to expand the action in slow fluctuations of the field $Q(\mathbf{x})$. The finite cost of these fluctuations is due to (a) their spatial variation and (b) the presence of the so-far neglected frequency mismatch, Ω. Fortunately, the job of actually determining the resulting contributions to the action has been done before [12]. All we need to do is carefully identify the parameters of the present problem with those of its electronic analogue; the algebraic structure of the two problems is almost identical. Indeed, substituting the 'soft' configurations $R = q_1 + \mathrm{i}q_2 Q$ into the action (39), we obtain

$$S[Q] = \mathrm{str}\ln\left[\frac{J}{2}\Delta + q_1 + \mathrm{i}q_2 Q + \frac{\Omega}{2\omega}\sigma_3^{AR}(q_1 + \mathrm{i}q_2 Q)\right]. \tag{45}$$

It is useful to relate this expression to the action of the fermionic problem (see [12])

$$S^{\mathrm{f}}[Q] = \mathrm{str}\ \ln\left(\frac{1}{2m}\Delta + E_F + \frac{\mathrm{i}}{2\tau}Q + \Omega\sigma_3^{\mathrm{AR}}\right). \tag{46}$$

Comparison of the two actions leads to the list of identifications summarized in table 1 below.

From (46), an *effective* low energy action

$$S^{\mathrm{f}}_{\mathrm{eff}}[Q] = \frac{\pi\nu}{8}\int \mathrm{d}^d r\ \mathrm{str}\left(D_0\partial Q\partial Q + 2\mathrm{i}\Omega Q\sigma_3^{\mathrm{AR}}\right). \tag{47}$$

can be derived by leading-order expansion in the parameters $\Omega\tau \ll 1$ and $l/L \ll 1$, where L is representative of the length scales we wish to probe. Equation (47) contains the bare values of DoS, ν, and diffusion constant, D_0, of the fermionic problem. These quantities are related to the parameters of the prototypical action, equation (46) as also summarized in the table 1. Details of the derivation can be found in [12].

Quite analogously, starting from the action equation (45), the effective low energy action of the magnon problem can be obtained as

$$S_{\mathrm{eff}}[Q] = \int \mathrm{d}^d r\ \mathrm{str}\left(\frac{\pi\Gamma^{(d)}}{(2\pi)^d}\frac{q_1^{\frac{d}{2}}}{8q_2}\left(\frac{2}{J}\right)^{\frac{d-2}{2}}\partial Q\partial Q + +\mathrm{i}\frac{q_1q_2}{\alpha\omega^3}\Omega Q\sigma_3^{\mathrm{AR}}\right). \tag{48}$$

We next proceed to discuss what can be learned from this representation of the problem. The first quantity we would like to calculate is the magnon density of states $\rho(\omega)$. This can be extracted from the correlation function C_1 with the help of equation (32) using the relations (41) and (43). The result is

$$\rho(\omega) = \frac{4q_1q_2}{\pi\alpha\omega^3}. \tag{49}$$

Using equation (49), we rewrite equation (48) in a form similar to the effective action of the fermion problem, equation (47):

$$S_{\text{eff}} = \frac{\pi\rho(\omega)}{8}\int \mathrm{d}^d r \left[D(\omega)\partial Q\partial Q + 2\mathrm{i}\Omega Q\sigma_3^{AR}\right], \tag{50}$$

Here $D(\omega)$ is the diffusion constant of magnons at frequency ω,

$$D(\omega) = \frac{\pi\Gamma^{(d)}}{(2\pi)^d}\left(\frac{2q_1}{J}\right)^{\frac{d-2}{2}}\frac{\alpha\omega^3}{4q_2^2}. \tag{51}$$

Finally, the product $\rho(\omega)D(\omega)$, relevant for the calculation of the thermal conductivity (cf equation (34) reads as

$$\rho(\omega)D(\omega) = \frac{\Gamma^{(d)}}{(2\pi)^d}\frac{q_1^{\frac{d}{2}}}{q_2}\left(\frac{2}{J}\right)^{\frac{d-2}{2}}. \tag{52}$$

Calculating the correlation function C_2 using equation (42), we indeed reproduce the expression equation (33).

Finally, we note that the dispersion relation of magnons is given by the poles of the Green function $G(\omega, \mathbf{p})$ and can be extracted from

$$\frac{J}{2}p^2 = q_1 + \mathrm{i}q_2. \tag{53}$$

Since the functional form of the mean field parameters $q_{1,2}$ depends on the dimensionality of space, we next consider the cases $d = 1, 2, 3$ separately.

4. Results

4.1. 1d case

Strictly one-dimensional Mattis glasses are amenable to transfer matrix techniques and have been discussed before us [3]. Nevertheless we would like to try to analyse them here using the techniques proposed in this paper. Our conclusion, however, is going to be that, as always for strictly one-dimensional problems, the sigma model description breaks down.

In $d = 1$, the momentum integral appearing in the saddle point equation (43) can readily be done and we obtain

$$\bar{R} = \left(\frac{\alpha^2\omega^4}{2J}\right)^{1/3}\cos(\pi/3)\left(1 + \mathrm{i}\tan(\pi/3)\sigma_3^{\mathrm{AR}}\right). \tag{54}$$

Substitution of this result into (49) leads to an SCBA DoS

$$\rho(\omega) = -\frac{1}{\pi}\mathrm{Im}(C_1(\omega)) \sim \omega^{-1/3}, \tag{55}$$

divergent at small frequencies. Equation (55) in fact coincides with the exact expression for the density of states obtained in [3] by the transfer matrix formalism. A direct application of SCBA to the 1D Mattis glass was attempted even earlier, in [8]. However, from a purist's point of view, the result does not seem quite trustworthy. A glance at (54) shows that the real (q_1) and the imaginary (q_2) part of the SCBA self-energy are actually of the same order. On the other hand, the SCBA as such is stabilized by the smallness of the parameter q_2/q_1. Put differently, in a one-dimensional environment the SCBA, and for that matter the construction of our low energy field theory, are ill-founded. For this reason, we are not going to discuss the case $d = 1$ any further and turn to $d = 2$ instead.

4.2. 2d case

In $d = 2$, the saddle point equation (43) takes the form

$$\bar{R} = \frac{\alpha\omega^2}{(2\pi)^2}\int \frac{\mathrm{d}^2 p}{\frac{J}{2}p^2 - \bar{R}}. \tag{56}$$

Cutting the logarithmic divergence of this integral by introducing a momentum cutoff Λ, we obtain

$$\bar{R} = \frac{\alpha\omega^2}{2\pi J}\log\left(-\frac{\Lambda^2 J}{2\bar{R}}\right).$$

To logarithmic accuracy, this equation is solved by $\bar{R} = q_1 + \mathrm{i}q_2\sigma_3^{\mathrm{AR}}$, where

$$q_1 = \frac{\alpha\omega^2}{2\pi J}\log\left(\frac{\pi\Lambda^2 J^2}{\alpha\omega^2}\right) \qquad q_2 = \frac{\alpha\omega^2}{2J}.$$

Notice that we are now on safe ground inasmuch as

$$q_1/q_2 \sim \log\left(\frac{\pi\Lambda^2 J^2}{\alpha\omega^2}\right) \gg 1, \tag{57}$$

or, in other words, as long as the frequency is small enough,

$$\omega \ll \frac{\Lambda J}{\sqrt{\alpha}}. \tag{58}$$

Armed with q_1 and q_2 we can now calculate the DoS, the diffusion constant and the thermal conductivity. With the help of equation (49) we find

$$\rho(\omega) = \frac{\alpha\omega}{\pi^2 J^2}\log\left[\frac{\pi\Lambda^2 J^2}{\alpha\omega^2}\right]. \tag{59}$$

This indeed coincides with the density of states derived in [8]. Looking at the dispersion relation equation (53) with the help of $q_2 \ll q_1$ we see that

$$\mathrm{Re}\,\omega = \sqrt{\frac{4\pi}{\alpha\log\left(\frac{\Lambda^2}{p^2}\right)}}\,\frac{J}{2}p, \qquad \mathrm{Im}\,\omega = \left(\frac{\pi}{\alpha^{\frac{1}{3}}\log\left(\frac{\Lambda^2}{p^2}\right)}\right)^{\frac{3}{2}}\frac{J}{2}p.$$

Next, the diffusion constant is given by

$$D(\omega) = \frac{J^2}{4\alpha\omega}. \tag{60}$$

This allows us to calculate the thermal conductivity by using equations (59), (60) and (34). Quite remarkably, the integral in equation (34) is convergent at small frequency and we obtain a closed formula for the magnon thermal conductivity in 2*d*

$$\kappa = \frac{k^2 T}{12\hbar}\log\left(\frac{\Lambda^2 J^2}{\alpha k^2 T^2}\right). \tag{61}$$

As a consequence of equation (57), this formula works if the logarithm in it is large, or $T \ll \Lambda J/k\sqrt{\alpha}$.

Finally, we would like to calculate the localization length of phonons at frequency ω. To do that, we evaluate the $\Omega = 0$ effective field theory action equation (48) to find

$$S_{\text{eff}} = \frac{1}{32\pi} \log\left(\frac{\pi J^2 \Lambda^2}{\alpha\omega^2}\right) \int \mathrm{d}^d r \,\mathrm{str}\, \partial Q \partial Q. \tag{62}$$

In two dimensions, the supersymmetric σ-model is known [12] to flow to a disordered phase. The magnon localization length is the length scale at which the corresponding RG group equations renormalize the coupling constant of the effective field theory,

$$\frac{1}{32\pi} \log\left[\frac{\pi \Lambda^2 J^2}{\alpha\omega^2}\right] \tag{63}$$

down to values of the order of 1. That gives (compare with [9])

$$l(\omega) \propto \exp\left\{\frac{1}{32\pi} \log\left[\frac{\pi \Lambda^2 J^2}{\alpha\omega^2}\right]\right\} \tag{64}$$

or

$$l(\omega) \propto \omega^{-\frac{1}{16\pi}}. \tag{65}$$

The localization length is divergent as a power law of the frequency. This is in contrast to the behaviour of phonons in structural glasses [9] and, by extension, of magnons in Mattis glass with nonzero overall magnetization. There the localization length diverges much faster with decreasing frequency, as an exponential of the inverse frequency square. In other words, in these systems, static disorder is less efficient as a scattering agent than in the Mattis glass with vanishing magnetization.

4.3. 3d case

Conceptually, the analysis of the $3d$ cause parallels the discussion of the previous section. We therefore restrict ourselves to a brief statement of the key formulae.

In analogy to the $2d$ case, we solve the mean field equations equation (43) to

$$q_1 = \frac{\alpha\omega^2}{\pi^2 J}\Lambda, \qquad q_2 = \sqrt{\frac{\alpha^3\omega^6\Lambda}{2\pi^4 J^4}}. \tag{66}$$

At sufficiently small frequencies ω

$$q_1/q_2 \propto \frac{J\sqrt{\Lambda}}{\omega\sqrt{\alpha}} \gg 1, \tag{67}$$

and the effective field theory approach works in this case as well. The DoS follows as

$$\rho(\omega) = 8\frac{(\alpha\Lambda)^{\frac{3}{2}}}{2^{\frac{3}{2}}\pi^5 J^3}\omega^2, \tag{68}$$

in agreement with the general mean field theory result above the critical dimension $d > d_c = 2$. The dispersion relation of magnons is

$$\mathrm{Re}\,\omega = \sqrt{\frac{\pi^2 J^2}{2\alpha\Lambda}}p \qquad \mathrm{Im}\,\omega = \frac{\pi}{4\Lambda}\sqrt{\frac{\pi^2 J^2}{2\alpha\Lambda}}p^2,$$

in agreement with the general result $\mathrm{Re}\,\omega \propto p$, $\mathrm{Im}\,\omega \propto p^{d-1}$. The diffusion constant is given by

$$D(\omega) = \frac{\pi^2}{6\sqrt{2}}\frac{J^3}{\sqrt{\Lambda}\alpha^{\frac{3}{2}}\omega^2}. \tag{69}$$

The thermal conductivity can be found with the help of equation (34) to give

$$\kappa = \frac{\Lambda}{9\pi}\frac{k^2 T}{\hbar}. \tag{70}$$

In $3d$, the sigma model at weak bare coupling $E_F\tau \sim q_1/q_2 \gg 1$ is known to flow to a metallic phase where the conductance $K \sim \kappa L$ is Ohmic. Accordingly, the heat conductance of a sample of linear size L will be given by

$$K = \frac{\Lambda L}{9\pi}\frac{k^2 T}{\hbar}. \tag{71}$$

Due to the condition, equation (67), the temperature has to satisfy

$$T \ll \frac{J}{k}\sqrt{\frac{\Lambda}{\alpha}}. \tag{72}$$

4.4. Quasi-1d case

Quasi-1d systems are highly anisotropic such that the extension in one direction (the 'longitudinal' direction) is far in excess of the 'transverse' extensions. Here we will consider such a three-dimensional 'wire' made of Mattis glass.

Let us denote the width of the wire, when measured in units of lattice spacings, N. Then the actual width of the wire is given by N/Λ, where Λ, as before, is the inverse lattice spacing. $N = 1$ corresponds to the strictly one-dimensional problem, while here we will consider the $N \gg 1$ case.

As a result of having finite N, the transverse momentum is quantized in units of (roughly) Λ/N. The saddle point equation (43) must now involve both integration over longitudinal momentum and the summation over the transverse ones. Fortunately it is possible to go back to three-dimensional integration if

$$q_2 \gg J\left(\frac{\Lambda}{N}\right)^2. \tag{73}$$

Once the integration is three-dimensional, we can borrow q_2 from equation (66) to find

$$J\sqrt{\frac{\Lambda}{\alpha}} \ll \omega N^{\frac{2}{3}}. \tag{74}$$

This is the condition which the frequency and the number of channels must satisfy for the $3d$ density of states be applicable to the quasi-1d geometry.

At the same time, for the formalism to work the frequency cannot be larger than $J\sqrt{\Lambda/\alpha}$, due to equation (67). We see that as a result, the frequency must lie in the interval

$$\frac{1}{N^{\frac{2}{3}}} \ll \frac{\omega}{J}\sqrt{\frac{\alpha}{\Lambda}} \ll 1, \tag{75}$$

which is always possible at large enough N.

Once the magnon frequency lies in the interval given by equation (75), the magnons are described by the $3d$ sigma model derived in the previous subsection. However, for sufficiently narrow wires and long times the diffusion of magnons becomes purely one-dimensional. For that to happen, we need to take the frequency Ω to be much less then the inverse Thouless time $D(\omega)/(N/\Lambda)^2$. In other words, using equation (69) we find

$$\Omega \ll \frac{\Lambda^{\frac{3}{2}} J^3}{N^2 \alpha^{\frac{3}{2}} \omega^2}. \tag{76}$$

Our conclusion is that in the long quasi-1d wires the magnons whose frequencies satisfy equation (75) diffuse and localize purely one-dimensionally for times longer than the inverse Ω given in equation (76) above, even though their density of states and their diffusion constant as a function of frequency ω are given by the 3d expressions (68) and (69).

5. Nonvanishing magnetization

5.1. Phonon–magnon crossover

It is also instructive to see how the magnons behave when the average magnetization is nonvanishing $M > 0$. While the properties of very low frequency magnons coincide with those of phonons in structural glasses discussed in [9], at somewhat higher frequencies the behaviour crosses over to the one indistinguishable from that of the zero magnetization magnons. It is this phenomenon that we would like to study in this section.

The starting point of the analysis is equation (28) with $M > 0$. It is fairly straightforward to repeat the analysis of section 3 of the paper to find the following effective action, the analogue of equation (45),

$$S[Q] = \mathrm{str}\ln\left[\frac{J}{2}\Delta + M\omega + q_1 + \mathrm{i}q_2 Q + \frac{\Omega}{2\omega}\sigma_3^{AR}(M\omega + q_1 + \mathrm{i}q_2 Q)\right]. \tag{77}$$

Accordingly, the self-consistent Born mean field equation (43) changes to

$$\bar{R} = \alpha\omega^2\left(\frac{1}{2\pi}\right)^d \int \frac{\mathrm{d}^d p}{\frac{J}{2}p^2 - \bar{R} - M\omega}. \tag{78}$$

The solution of this equation has, as before, the form

$$\bar{R} = q_1 + q_2\sigma^{AR}. \tag{79}$$

Finally, the structural form of Green functions and observables themselves changes. Specifically, the Green functions now have the form

$$G^{\pm} = \left[\omega(V + M) + \frac{J}{2}\Delta \pm \mathrm{i}\epsilon\,\mathrm{sign}\,\omega\right]^{-1}(V + M). \tag{80}$$

Comparison with the fermionic effective action equation (46) shows that the role of the Fermi energy is now played by $M\omega + q_1$. Two distinct regimes can now be identified: for $M \ll q_1(\omega)/\omega$, the magnon properties are indistinguishable from those of Mattis glass magnons in the absence of a magnetized background. In contrast, for $M \gg q_1(\omega)/\omega$, the magnons resemble phonons in structural glasses. As we shall see in a moment, at sufficiently low frequencies, the second scenario is effectively realized.

To demonstrate how this works, let us solve the SCBA equation (equation (78)) at $d = 2$. We find

$$q_1 = \frac{\alpha\omega^2}{2\pi J}\log\left(\frac{\Lambda^2 J}{2q_1 + 2M\omega}\right), \qquad q_2 = \frac{\alpha\omega^2}{2J}. \tag{81}$$

If

$$\frac{\alpha\omega^2}{2\pi J}\log\left(\frac{\Lambda^2 J}{2M\omega}\right) \ll M\omega, \tag{82}$$

the magnons are in the phonon regime. This will happen if

$$\omega \ll \frac{JM}{\alpha}. \tag{83}$$

Conversely, at $\omega \gg JM/\alpha$, the magnons are effectively in the $M = 0$ regime studied in this paper in the previous section. However, the condition $q_1 \gg q_2$ must also apply in order for the formalism developed in this paper to work. This is equivalent to $\omega \ll \Lambda J/\sqrt{\alpha}$. The consistency condition is thus $\Lambda J/\sqrt{\alpha} \gg JM/\alpha$ or

$$M \ll \sqrt{\Lambda^2 \alpha}. \tag{84}$$

The average magnetization $M \leqslant 1$, while $\Lambda^2\alpha > 1$. We see that equation (84) always applies.

Analogously, at $d = 3$, we find that the magnetization M can be neglected if

$$\omega \gg \frac{MJ}{\alpha\Lambda}. \tag{85}$$

Its consistency with $q_1 \gg q_2$ gives $M \ll \sqrt{\Lambda^3\alpha}$.

5.2. The phonon regime

At small enough frequency, the magnons behave in a way reminiscent of phonons in structural glasses. Even though most of the results for those were obtained in [9] we would like to present some of them here, in part for completeness, and in part since there are still some distinctions between phonons and magnons. This mostly stems from the fact that the basic equation for phonons in glasses, while practically identical with equation (28), had ω^2 in place of ω.

The main feature simplifying the magnetized theory at small frequency is that M always dominates the effective 'Fermi' energy. As a result, it is possible to neglect q_1 in all the calculations. The Green functions can be approximately calculated as

$$G^{\pm} \approx M \left[\omega(V + M) + \frac{J}{2}\Delta \pm \mathrm{i}\epsilon \,\mathrm{sign}\,\omega \right]^{-1}. \tag{86}$$

Because of this structure, the problem enjoys a complete analogy to the problem of disordered fermions, and the results of the treatment of the latter can be directly translated into the language of the former. This fact has already been successfully exploited in [9].

In particular, it is clear that the DoS will coincide with what we called ν in the previous sections of the paper,

$$\rho(\omega) = \nu(\omega) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d p\, \delta\left(\frac{J}{2}p^2 - M\omega\right) = \frac{\Gamma^{(d)}}{(2\pi)^d}\frac{d}{J}\left(\frac{2M\omega}{J}\right)^{\frac{d-2}{2}}. \tag{87}$$

This coincides with the spectrum of ferrimagnets [1, 4].

Concentrating on the case of two dimensions $d = 2$, we find that as before $q_2 = \alpha\omega^2/(2J)$, which leads to the diffusion constant

$$D(\omega) \equiv D_0(\omega) = \frac{2J^2M}{\alpha\omega}. \tag{88}$$

The integral $\int \mathrm{d}\omega\, \rho(\omega) D(\omega)$ is logarithmically divergent, leading to the infinite thermal conductivity of magnons at nonzero magnetization. A similar phenomenon is also observed with phonons in structural glasses (although the degree of divergence is different there). The physical consequence of this latter phenomenon is that in real magnets with nonvanishing magnetization the thermal conductivity must be dominated by inelastic scattering processes (completely neglected in this paper) rather than by elastic collisions.

We finally note that the dispersion relation can be deduced from $Jp^2/2 = M\omega + \mathrm{i}q_2(\omega)$. In $2d$ this can be reduced to $\mathrm{Im}\,\omega \propto p^4$. Similarly, in $3d q_2 \propto \omega^{\frac{5}{2}}$ resulting in $\mathrm{Im}\,\omega \propto p^5$. This confirms equation (10). Incidentally, equation (9) also follows from these considerations, but with the substitution $\omega \to \omega^2$. This follows from the fact that equation (28) basically coincides with the equation of motion of phonons in disordered solids, discussed in [9], up to this change of the definition of frequency.

The diffusion constant goes as $D \equiv D_0(\omega) \propto 1/\omega^{3/2}$, leading to the same divergent behaviour of $D(\omega)\rho(\omega) \propto 1/\omega$ as in $2d$.

6. Conclusions

In this paper we studied the spectral and transport properties of magnons in a Mattis glass with vanishing average magnetization. We find that in $3d$ their motion is diffusive and that results in finite thermal conductivity given by equation (70). In $2d$ the thermal conductance is also finite and given by equation (61). This could be contrasted with the behaviour of phonons in structural glasses (and with magnons in a Mattis glass with nonzero magnetization), whose contribution to thermal conductance is infinite (when phonon–phonon scattering is neglected).

The prime motivation for applying the field theoretical formalism above to the Mattis glass is that in the latter the two problems of identifying the ground states of the system and quantifying the low-lying excitations superimposed on these ground states (*the* two fundamental issues in understanding the behaviour of glassy systems) afford a clear separation. What will happen in 'real' glasses? Although we are not in a position to say anything quantitative, one may expect the excitations on top of the—now unknown—ground state(s) to be governed by *some* quadratic bosonic Hamiltonian (cf equation (18).) Assuming this Hamiltonian to be (a) positive, (b) local and (c) reflecting the presence of an underlying Goldstone mode, significant elements of the present analysis are likely to remain valid. The quantitative formulation of this analysis will be the subject of future research.

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Appendix

We would like to show that

$$\left[\omega V + \frac{J}{2}\Delta \pm \mathrm{i}\epsilon V\right]^{-1} = \left[\omega V + \frac{J}{2}\Delta \pm \mathrm{i}\epsilon \,\mathrm{sign}\,\omega\right]^{-1} \tag{A.1}$$

in the limit when ϵ is small. Once this is shown, the expression for the Green function equation (30) follows immediately.

The proof is based on the fact that $-\frac{J}{2}\Delta$ is a positive definite operator. For simplicity, let us first consider an analogue of equation (A.1) written for real numbers as opposed to operators. Take

$$\frac{1}{\omega v - d \pm \mathrm{i}v\epsilon}, \tag{A.2}$$

where v is a real number and d is a positive real number. Take $\omega > 0$. Then if v is negative, $\omega v - d$ is also negative, and ϵ can be completely neglected. If v is positive, the product $v\epsilon$

can be replaced by ϵ, since we are interested in the limit $\epsilon \to 0$ in any case. In other words, regardless of the sign of v,

$$\frac{1}{\omega v - d \pm \mathrm{i}v\epsilon} = \frac{1}{\omega v - d \pm \mathrm{i}\epsilon}, \qquad \omega > 0 \tag{A.3}$$

as ϵ is taken to zero. For negative ω, the logic can be repeated to result in

$$\frac{1}{\omega v - d \pm \mathrm{i}v\epsilon} = \frac{1}{\omega v - d \mp \mathrm{i}\epsilon}, \qquad \omega < 0. \tag{A.4}$$

These can be combined to give

$$\frac{1}{\omega v - d \pm \mathrm{i}v\epsilon} = \frac{1}{\omega v - d \pm \mathrm{i}\epsilon \operatorname{sign}\omega}, \tag{A.5}$$

at arbitrary sign of ω.

The operator generalization of equation (A.5) can be derived by similar reasoning: take $\omega > 0$ as in the text above. To show that equation (A.1) holds, we calculate matrix elements of

$$G^{-1} \equiv \omega V + \frac{J}{2}\Delta \pm \mathrm{i}\epsilon V \tag{A.6}$$

in the basis where $\omega V + \frac{J}{2}\Delta$ is diagonal. Denoting the eigenvalues of $\omega V + \frac{J}{2}\Delta$ as λ_a and the eigenvalues of $\frac{J}{2}\Delta$ as $-\mu_a$ ($\mu_a > 0$), we find

$$\left(\omega V + \frac{J}{2}\Delta \pm \mathrm{i}\epsilon V\right)_{ab} = \lambda_a\delta_{ab} \pm \frac{\mathrm{i}\epsilon}{\omega}\left(\lambda_a\delta_{ab} + U_{ac}\mu_c U^\dagger_{cb}\right), \tag{A.7}$$

where the matrix elements of $\frac{J}{2}\Delta$ in the reference basis are written as $-U_{ac}\mu_c U^\dagger_{cb}$, U is some unitary matrix and summation over the index c is implied. In what follows, it is crucial that $U_{ac}\mu_c U^\dagger_{ca} > 0$, or in other words, that the diagonal entries of a positive matrix have to be positive.

The eigenvalues λ_a are generally nonzero. As we tune the parameter ω, the eigenvalues λ_a can go through zero one at a time. If all of λ_a are nonzero, ϵ can be taken to zero directly in equation (A.7), and we obtain

$$G_{ab} = \frac{\delta_{ab}}{\lambda_a}. \tag{A.8}$$

If one of λ_a is zero, more care is needed. The off-diagonal matrix elements of G can be found as the ratio of the appropriate minors of G^{-1}_{ab} to the determinant of G^{-1}_{ab}. It is easy to see that the determinant of G^{-1}_{ab} is of the order of ϵ, while the minors are at least of the order of ϵ as well (or they could be of higher order in ϵ). As a result, the off-diagonal matrix elements of G_{ab} are either constant as ϵ goes to zero, or vanishing. However, being a constant for a particular value of ω (for which one of the eigenvalues λ_a vanished) and zero at all other values of ω is tantamount to being zero (as in $\lim_{\epsilon\to 0} \mathrm{i}\epsilon/(\lambda + \mathrm{i}\epsilon) = 0$ as a function of λ).

As far as the diagonal matrix elements of G are concerned, they are also given by the ratio of the minors to the determinant of G^{-1}. The main contribution to both in the limit $\epsilon \to 0$ is the product of the appropriate diagonal elements of G^{-1}. It is easy to see that in that limit the matrix elements of G_{aa} are equal to $1/\lambda_a$ except when $\lambda_a = 0$. For that one vanishing eigenvalue,

$$G_{aa} = \frac{\omega}{\mathrm{i}\epsilon\left(U_{ac}\mu_c U^\dagger_{ca}\right)}.$$

Since $U_{ac}\mu_c U^\dagger_{ca} > 0$, and $\omega > 0$, and since ϵ is taken to zero, we can replace this by

$$G_{aa} = \frac{1}{\mathrm{i}\epsilon}$$

in that limit. So indeed, the coefficient in front of ϵ can simply be put to 1 from the very beginning, as in equation (A.1) for $\omega > 0$.

Repeating the same argument for $\omega < 0$, we arrive at equation (A.1) for arbitrary sign of ω.

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